UNIVERSIDAD TÉCNICA DE MANABÍ
Fundada en 1952

BASES DE LA CIENCIA
REVISTA CIENTÍFICA
FACULTAD DE CIENCIAS BÁSICAS

EDICIÓN ESPECIAL - ISSN 2588-0764

Publicación Cuatrimestral. Vol. 7, No Especial, Diciembre, 2022, Ecuador (p.107-121). Edición continua
https://revistas.utm.edu.ec/index.php/Basedelaciencia/index
revista.bdlaciencia@utm.edu.ec
Universidad Técnica de Manabí
DOI: https://doi.org/10.33936/revbasdelaciencia.v7iESPECIAL.4722

Ciencias Matemáticas

UN ESTUDIO EXPLORATORIO SOBRE TOPOLOGÍAS PRIMALES

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Recibido: 26-05-2022/ Aceptado: 12-12-2022 / Publicación: 26-12-2022

Editor Académico: Luis Bladismir Ruiz Leal

RESUMEN

Sea X un conjunto no vacío y $f: X \to X$ una función. La colección $\tau_f := \{A \subset X: f^{-1}(A) \subset A\}$ es llamada topología primal inducida por f sobre X. Con esta topología, el espacio X es un espacio Alexandroff, es decir, la intersección de una familia arbitraria de abiertos es un conjunto abierto. Una topología τ provista de dos operaciones biarias definidas a través de la unión e intersección de conjuntos puede ser vista como un semianillo. El objetivo de este trabajo es mostrar algunas de las propiedades generales de las topologías primales, en particular, las características de las topologías primales vistas como semianillos. Entre otros resultados, proveemos algunos de los ideales primos y maximales que se pueden contruir para una topología primal arbitraria. Finalmente, proveemos algunos resultados relacionados a las topologías primales inducidas por un homomorfismo de grupos sobre un grupo G.

Palabras clave: Espacio primal, semianillo, homomorfismo de grupo.

AN EXPLORATORY STUDY ON PRIMAL TOPOLOGIES

ABSTRACT

Let X be a non-empty set and $f: X \to X$ a function. The collection $\tau_f := \{A \subset X: f^{-1}(A) \subset A\}$ is called primal topology induced by f on X. With this topology, the space X is an Alexandroff space, that is, the intersection of an arbitrary family of open sets is an open set. A topology τ equipped with two binary operations defined through the union and intersection of sets can be seen as a semiring. The aim of this paper is to show some of the general properties of primal topologies, in particular, the characteristics of primal topologies seen as semirings. Among other results, we provide some of the prime and maximal ideals that can be constructed for an arbitrary primal topology. We finally provide some results related to the primal topology induced by a group homomorphism on a group G.

Keywords: Primal space, semiring, group homomorphism.

UM ESTUDO EXPLORATÓRIO SOBRE TOPOLOGIAS PRIMAIS

RESUMO

Seja X um conjunto não vazio e $f: X \to X$ uma função. A coleção $\tau_f \coloneqq \{A \subset X: f^{-1}(A) \subset A\}$ é chamada de topologia primal induzida por f em X. Com esta topologia, o espaço X é um espaço de Alexandroff, ou seja, a interseção de uma família arbitrária de conjuntos abertos é um conjunto aberto. Uma topologia τ provida de duas operações binárias definidas por união e interseção de conjuntos pode ser vista como um semi-anel. O objetivo deste artigo é mostrar algumas das propriedades gerais das topologias primais, em particular, as características das topologias primais vistas como semi-anéis. Entre outros resultados, fornecemos alguns dos ideais primos e máximos que podem ser construídos para uma topologia primal arbitrária. Finalmente, fornecemos alguns resultados relacionados à topologia primal induzida por um homomorfismo de grupo em um grupo G.

Palavras chave: espaço primal, semi-anel, homomorfismo de grupo

Citación sugerida: Garcia-Mendoza, C., Vielma, J., Lasser, J. (2022). Un estudio exploratorio sobre topologías primales. Revista Bases de la Ciencia, 7 (No Especial), Diciembre, 107-121. DOI: https://doi.org/10.33936/revbasdelaciencia.v7i3.4722

1. INTRODUCTION

Alexandroff (1937) introduced the notion of topological spaces in which the arbitrary union of closed sets is a closed set, which he named *Diskrete Raume* or Discrete Spaces. This property is clearly equivalent to the fact that the arbitrary intersection of open sets is an open set. A trivial example of these spaces, on which mathematicians such as McCord (1966), Stong (1966) and Herman (1990) worked on, correspond to finite topological spaces. However, given that the term *discrete space* was already being used to describe those on which every subset is open, McCord, for instance, decided to rename them as *A-spaces*, and focused his study on T_0 A-spaces. Herman (1990) on the other hand, named these spaces *Sparse*, but mathematicians would eventually opt for the name Alexandroff spaces in honor of Pavel Alexandroff, whom initially drew the attention to this topic.

Shirazi and Golestani (2011) would later present a study on a proper subclass of Alexandroff spaces, which they called *Functional Alexandroff Spaces*, given that the topology considered was induced by a function, and the resultant space was indeed Alexandroff. Finally, Echi (2012) presented some results about these spaces and named them *Primal spaces*, term that has been used since. In this paper, we equip a primal topology τ_f with two operations defined through the union and intersections of sets, which make τ_f a semiring. We present some of the properties of these topologies seen as semirings, more specifically, we show some of the prime and maximal ideals that can be constructed for a primal topology τ_f. We also study the generalization of some results previously presented by the authors of this paper. In particular, we show the generalization of primal spaces induced on a finite dimension vector space by the use of group homomorphisms.

2. PRELIMINARIES

In this section we present some of the fundamental concepts about primal topologies.

Definition 2.1. *Given a non-empty set* X *and a function* $f: X \to X$, *then the collection* $\tau_f \coloneqq \{A \subset X: f^{-1}(A) \subset A\}$ *is called primal topology induced on* X, *and the space* (X, τ_f) *is called primal space.*

Equivalently, a primal topology τ_f on a set X can be defined by deciding the closed sets to be those subsets $B \subset X$ that are f- invariant, that is, $f(B) \subset B$. In order to see this equivalence, let A be an open set of a primal space (X, τ_f), then by definition $X \setminus A$ is closed. If we assume that the image of $X \setminus A$ under f is not contained in $X \setminus A$, then there exists an element $x \in A$ such that $f^{-1}(x) \in X \setminus$

A, a contradiction. Some of the well-known properties of the function associated to a primal space are given below.

Lemma 2.1. *The function f associated to the primal space (X, τ_f) is continuous.*

Proof: Let A be an open set of X. In order for $f^{-1}(A)$ to be open, it must hold $f^{-1}(f^{-1}(A)) \subset f^{-1}(A)$. Let $x \in f^{-1}(A)$. Since A is open, we have $f^{-1}(A) \subset A$, which implies $x \in A$. By the same hypothesis we then have $f^{-1}(x) \in f^{-1}(A)$.

The following two Lemmas are provided by Shirazi and Golestani (2011), for which we provide the proofs.

Lemma 2.2. *Let (X, τ_f) be a primal space. Then $f: X \to X$ is a homeomorphism if and only if it is a bijection.*

Proof: If f is a homeomorphism, it is trivial that f is a bijection. For the converse statement, by Lemma 2.1 we have that f is continuous. In order for f^{-1} to be continuous, it must hold $f^{-1}(f(A)) \subset f(A)$, for $A \in \tau_f$. Given that f in injective, then $f^{-1}(f(A)) = A$, which implies that $A \subset f(A)$ must hold. This is true given that $f(A) = f(f^{-1}(A) \cup [A \setminus f^{-1}(A)]) = f(f^{-1}(A)) \cup f([A \setminus f^{-1}(A)]) = A \cup f([A \setminus f^{-1}(A)]) \supset A$.

Lemma 2.3. *Let (X, τ_f) be a primal space. Then $f: X \to X$ is a homeomorphism if and only if it is injective and open.*

Proof: If f is a homeomorphism, it is trivial that it is an injective and open function. For the converse statement, by Lemma 2.1 we have that f is continuous. For f^{-1} to be continuous, it must hold that $f(A) \in \tau_f$, for $A \in \tau_f$, which is true given that f is open.

We shall now prove that primal topologies are indeed Alexandroff topologies. In order to do that, we first define Alexandroff spaces and provide one of its characterizations.

Definition 2.2. *Let X be a topological space, then X is an Alexandroff space if the arbitrary intersection of open sets is an open set.*

The following theorem, which characterizes Alexandroff spaces, is given by Speer (2007).

Theorem 2.1. *X is an Alexandroff space if and only if every point $x \in X$ has a minimal open neighborhood.*

We resort to this result in order to prove that every primal space is Alexandroff. For this reason, we define a set for every point in a primal space which will be proven to be the minimal open neighborhood of the point.

Definition 2.3. *Let (X, τ_f) be a primal space and $x, y \in X$. A preorder $\leq_f$ can be defined on X as follows: $x \leq_f y$ if and only if there exists an integer $n \geq 0$ such that $f^n(x) = y$.*

Definition 2.4. *Let (X, τ_f) be a primal space. Then the sets:*

$$cl(x) = \{y \in X: x \leq_f y\} \quad and \quad ker(x) = \{y \in X: y \leq_f x\}$$

are called the closure and kernel of x respectively.

Proposition 2.1. *Let (X, τ_f) be a primal space and $x \in X$, then $ker\ (x)$ is the minimal open set containing x.*

Proof: We assume there exists an open set $A \in \tau_f$ such that $x \in A \subsetneq \ker\ (x)$. Then there exists an element $y \in (\ker(x) \setminus A)$. Moreover, there exists an integer $k \geq 0$ such that $f^k(y) \notin A$ and $f^{k+1}(y) \in A$, which implies $f^{-1}(A) \not\subset A$, a contradiction.

Corollary 2.1. *Every primal space is an Alexandroff space.*

Proof: It is an immediate result from Theorem 2.1 and Proposition 2.1.

We now show some of the fundamental properties of the sets defined above.

Lemma 2.4. *Let x, y be two distinct elements of a primal space (X, τ_f). Then $ker(x) \subset ker\ (y)$ or $ker(y) \subset ker\ (x)$ or $ker(x) \cap ker(y) = \emptyset$.*

Proof: We assume $\ker(x) \cap \ker(y) \neq \emptyset$. Then there exists an element $z \in X$ and integers $n, m \geq 0$ such that $f^n(z) = x$ and $f^m(z) = y$. Given that $x \neq y$ we have that $n \neq m$. If we assume $m > n$ then $x \in \ker\ (y)$ and $\ker(x) \subset \ker\ (y)$. If we assume $n > m$ then $y \in \ker(x)$ and $\ker(y) \subset \ker(x)$.

Lemma 2.5. *Let (X, τ_f) be a primal space. Then for all $x \in X$ we have $cl(x)$ is the minimal closed set containing x.*

Proof: We assume there exists a closed set A of X such that $x \in A \subsetneq cl(x)$. Then there exists an element $y \in (cl(x) \setminus A)$ and an integer $k \geq 0$ such that $f^k(x) \in A$ and $f^{k+1}(x) = y$, which implies $f(A) \not\subset A$, a contradiction.

Therefore, any closed set A of X can be expressed as a union of the closure of each element of A. More precisely, if A is a closed set of X then $A = \bigcup_{x\in A} cl(x)$. Observe that given that every primal space is Alexandroff (equivalent to the fact that the arbitrary union of closed sets is a closed set), we have that $\bigcup_{x\in A} cl(x)$ is indeed a closed set.

Lemma 2.6. *Let (X, τ_f) be a primal space and A an open subset of X, then A is a primal space equipped with the subspace topology:*

$$\tau_A \coloneqq \{A \cap U : U \in \tau_f\}$$

Proof: Let B be an element of τ_A, then $B = A \cap U$ for some $U \in \tau_f$. Also $f^{-1}(B) = f^{-1}(A \cap U) = f^{-1}(A) \cap f^{-1}(U)$ and given that A and U are open set of X it follows $f^{-1}(A) \cap f^{-1}(U) \subset (A \cap U) = B$. Therefore $f^{-1}(B) \subset B$ and τ_A is a primal topology.

In regards of the connected components of a primal space, Shirazi and Golestani (2011) show the following result.

Lemma 2.7. *Let (X, τ_f) be a primal space, then any two elements $p, q \in X$ are in the same connected component if and only if there exist $n, m \in \mathbb{N}$ such that $f^n(p) = f^m(q)$.*

Given this Lemma, it is possible to define an equivalence relation $\sim_f$ for (X, τ_f) as follows: for any two elements $x, y \in X$, $x \sim_f y$ if and only if there exist integers $n, m \geq 0$ such that $f^n(x) = f^m(y)$.

Proposition 2.2. *The relation $\sim_f$ is indeed an equivalence relation.*

Proof: Reflexivity and symmetry are trivial. To prove transitivity, let $x \sim_f y$ and $y \sim_f z$, then there exist integers $n, m, j, k \geq 0$ such that $f^m(x) = f^n(y)$ and $f^j(y) = f^k(z)$. Therefore $f^{m+j}(x) = f^j(f^m(x)) = f^j(f^n(y)) = f^n\left(f^j(y)\right) = f^n\left(f^k(z)\right) = f^{n+k}(z)$, and $x \sim_f z$.

Naturally, it is possible to define the equivalence class of each element $x \in X$ as the set $\tilde{x} \coloneqq \{y \in X : x \sim_f y\}$ and the quotient space as the set $\frac{X}{\sim_f} \coloneqq \{\tilde{x} : x \in X\}$

3. GENERAL RESULTS ABOUT PRIMAL SPACES

In this section we present some general results about primal spaces. In particular, we focus on some of the characteristics of connected components in primal spaces.

Lemma 3.1. *Let (X, τ_f) be a primal space and $y \in \tilde{x}$, then $cl(y) \subset \tilde{x}$.*

Proof: Let $z \in cl(y)$, then there exists an integer $n \geq 0$ such that $f^n(y) = z$. Given that $y \in \tilde{x}$ then there exist integers $m, l \geq 0$ such that $f^m(x) = f^l(y)$. Applying the result from Proposition 2.2 we have $f^{m+n}(x) = f^l(z)$ which implies $x \sim_f z$ and $z \in \tilde{x}$.

Given this result, it is possible to show that every connected component of a primal space is closed. This result is also shown by Garcia-Mendoza et al (2021), however we provide a different approach to prove such property.

Lemma 3.2. *Let (X, τ_f) be a primal space, then $\tilde{x} = \bigcup_{y \in \tilde{x}} cl(y)$.*

Proof: Let $y \in \tilde{x}$, since $y \in cl(y)$ and $cl(y) \subset \bigcup_{y \in \tilde{x}} cl(y)$ we have $\tilde{x} \subset \bigcup_{y \in \tilde{x}} cl(y)$. On the other hand, let $y \in \bigcup_{y \in \tilde{x}} cl(y)$, then there exists $z \in \tilde{x}$ such that $y \in cl(z)$ and by Lemma 3.1 we have $cl(z) \subset \tilde{x}$, which implies $\bigcup_{y \in \tilde{x}} cl(y) \subset \tilde{x}$.

Note that every connected component is clopen, since the equivalence classes form a partition of the space X, and the complement of each connected component is the union of closed connected components. The following result is shown by Guale et al (2020).

Lemma 3.3. *Let (X, τ_f) be a connected primal space and let $A, B \in \tau_f$. If $A \cup B = X$, then $A = X$ or $B = X$.*

We propose the following generalization of the previous Lemma as follows.

Lemma 3.4. *Let (X, τ_f) be a connected primal space such that $cl(x)$ is finite for every $x \in X$. Let $\{A_\alpha : \alpha \in J\}$ be a collection of open sets of X such that $\bigcup_{\alpha \in J} A_\alpha = X$. Then there exists $A_0 \in \{A_\alpha : \alpha \in J\}$ such that $A_0 = X$.*

Proof: If X is a connected primal space and $cl(x)$ is finite for all $x \in X$ then there exists $p \in X$ such that p is a periodic point. It must hold that $p \in A_0$ for some $A_0 \in \{A_\alpha : \alpha \in J\}$. It also holds that $p \in \ker(p) = X$, which implies $A_0 = X$.

Observe that this result cannot be extended for the case where $cl(x)$ is infinite for all x is a connected primal space. In order to see this, it is necessary to introduce the following equivalence.

Lemma 3.5. *Let (X, τ_f) be a connected primal space, then the following are equivalent:*

a. *If $X = \bigcup_{\alpha \in J} A_\alpha$ for open sets A_α of X, then there exists $j \in J$ such that $A_j = X$*
b. *There exists $x \in X$ such that $ker(x) = X$*

Proof: a → b: Consider the following union $\bigcup_{x\in X} \ker(x)$ which is clearly equal to X. From a. it follows that there exists $x \in X$ such that $\ker(x) = X$. b → a: Consider a collection of open sets A_α with $\alpha \in J$ such that $\bigcup_{\alpha\in J} A_\alpha = X$. By b. it follows that there exists $x \in X$ such that $\ker(x) = X$. By Proposition 2.1 we have that $x \in \ker(x) \subset A_j$ for some $j \in J$.

Observe that this equivalence implies the existence of a periodic point $x \in X$. To prove this, assume proposition b. from Lemma 3.5 as true. Let $x \in X$ such that $\ker(x) = X$, then $f(x)$ is also an element of X and by b. it is also an element of $\ker(x)$, which implies that there exists an integer $n \geq 0$ such that $f^n(f(x)) = f^{n+1}(x) = x$, therefore x is a periodic point. The existence of a periodic point impedes the existence of an infinite $cl(y)$ for some y in a connected primal space X. In order to illustrate this, consider the following trivial example.

Example 3.1. *Let $\mathbb{N}$ be the set of natural numbers and $f: \mathbb{N} \to \mathbb{N}$ a function defined by $f(n) = n + 1$. Then $(\mathbb{N}, \tau_f)$ is a connected primal space and $cl(n)$ is infinite for all $n \in \mathbb{N}$.*

If we assume there exists a collection of open sets $\{A_\alpha : \alpha \in J\}$ such that $\mathbb{N} = \bigcup_{\alpha\in J} A_\alpha$, then by the equivalence shown in Lemma 3.5 it must hold that there exists $n \in \mathbb{N}$ such that $\ker(n) = \{x \in \mathbb{N} : x \leq n\} = \mathbb{N}$, a contradiction.

4. PRIMAL TOPOLOGIES SEEN AS SEMIRINGS

In this section we study some of the properties of primal topologies seen as semirings. In particular, we show some of the prime ideals that can be constructed for an arbitrary primal topology.

Definition 4.1. *A semiring is a set X equipped with two binary operations $+$ and $\cdot$, called addition and multiplication respectively, such that:*

- *$(X, +)$ is a commutative monoid with identity element 0*
- *$(X, \cdot)$ is a monoid with identity element 1*
- *Multiplication distributes over addition*
- *Multiplication by 0 annihilates X.*

Example 4.1. *The set of all square matrices $n \times n$ with positive entries with the usual addition and multiplication between matrices is a semiring.*

Lemma 4.1. *If (X, τ) is a topological space, then τ is a semiring with $A + B := A \cup B$ and $A \cdot B := A \cap B$ as the addition and multiplication operations.*

Proof: The union and intersection of sets are clearly binary operations. Moreover, the identity element of the addition and multiplication operations are $\emptyset$ and X respectively. The commutativity, associativity and distributity of these operations are obtained from the properties of union and intersection of sets. Finally, it is clear that multiplication by $\emptyset$ annihilates τ.

Given that the multiplication operation is defined through the intersection of sets, a commutative binary operation, we have that a topology τ is actually a commutative semiring, equipped with the operations above defined.

Definition 4.2. *A subset I of a semiring $(S,+,\cdot)$ is called an ideal of S, if the identity element of the addition operation is an element of I, and for every $a,b \in I$ and $s \in S$ it holds $a + b \cdot s \in I$.*

Definition 4.3. *A semiring homomorphism from a semiring S to a semiring R is a function $f: S \to R$ such that for all $a,b \in S$ it holds:*

5. $f(a +_S b) = f(a) +_R f(b)$
5. $f(a \cdot_S b) = f(a) \cdot_R f(b)$
5. $f(1_S) = 1_R$

Lemma 4.2. *Let $f: S \to R$ be a semiring homomorphism and let I be an ideal of R, then $f^{-1}(I) = \{a \in S: f(a) \in I\}$ is an ideal of S.*

Proof: It is easy to see that $0_S \in f^{-1}(I)$. Let $a,b \in f^{-1}(I)$, then it holds $f(a), f(b) \in I$, and given that I is an ideal, then $f(a) + f(b) \in I$. Given that f is a semiring homomorphism, then $f(a + b) = f(a) + f(b) \in I$, which implies $a + b \in f^{-1}(I)$. Using similar arguments it can be seen that $sa \in f^{-1}(I)$ and $f^{-1}(I)$ is an ideal of S.

Definition 4.4. *A proper ideal P of a semiring $(S,+,\cdot)$ is called a prime ideal of S, if $ab \in P$ implies $a \in P$ or $b \in P$.*

Definition 4.5. *A proper ideal M of a semiring $(A,+,\cdot)$ is called maximal ideal of S if $M \subseteq I \subseteq S$, for an ideal I of S, then $M = I$ or $I = S$.*

Note that for an ideal I to be a proper ideal of a semiring S it must hold that $I \subsetneq S$, which in turn requires $X \notin I$.

Lemma 4.3. *Let (X,τ) be a topological space, then $\sigma(x) = \{A \in \tau: x \notin A\}$ is a prime ideal of τ.*

Proof: It easy to see that $\emptyset \in \sigma(x)$. Let $A,B \in \sigma(x)$ and $V \in \tau$. Given that $x \notin A$ and $x \notin B$ then $x \notin (A \cup B)$ which implies $(A \cup B) \in \sigma(x)$. It is trivial that $(A \cap V) \in \sigma(x)$. If $P \cap Q \in \sigma(x)$ for $P, Q \in$

τ then $P \in \sigma(x)$ or $Q \in \sigma(x)$. Indeed, if we assume $x \in P$ and $x \in Q$ then $x \in (P \cap Q)$ and $(P \cap Q) \notin \sigma(x)$, a contradiction. It must hold then that $x \notin P$ or $x \notin Q$, that is $P \in \sigma(x)$ or $Q \in \sigma(x)$.

Theorem 4.1. *If (X, τ) is a T_1 topological space, then $\sigma(x) = \{A \in \tau : x \notin A\}$ is a maximal ideal of τ.*

Proof: Given that X is T_1, then $X \setminus \{x\}$ is open, moreover $(X \setminus \{x\}) \in \sigma(x)$. If we assume $\sigma(x)$ is not a maximal ideal of τ, then there exists an ideal I of τ such that $\sigma(x) \subset I \subseteq \tau$. Therefore, there exists an open set $A \in I$ such that $x \in A$. Then $(X \setminus \{x\}) \cup A = X \in I$, which implies $I = \tau$. It follows that $\sigma(x)$ is a maximal ideal of τ.

Theorem 4.2. *Let (X, τ_f) be a primal space and C a connected component of X. Then $\psi(C) \coloneqq \{A \in \tau_f : C \not\subset A\}$ is a prime ideal of τ_f.*

Proof: It is easy to see that $\emptyset \in \psi(C)$. Let $A, B \in \psi(C)$, $V \in \tau_f$ and C a connected component of X. If $(A \cup B) \supset C$, then applying Lemma 2.6 and Lemma 3.3 we have $C \subset A$ or $B \subset C$, a contradiction. Therefore $(A \cup B) \not\supset C$ and $(A \cup B) \in \psi(C)$. It is trivial that $(V \cap A) \in \psi(C)$. Finally, if $P \cap Q \in \psi(C)$ for $P, Q \in \tau_f$, then $P \in \psi(C)$ or $Q \in \psi(C)$. Indeed, if we assume $P \notin \psi(C)$ then $C \subset P$, and given that $(P \cap Q) \not\supset C$, it must hold that $Q \not\supset C$, that is $Q \in \psi(C)$. A similar result is obtained if we assume $Q \notin \psi(C)$.

Garcia-Mendoza et al (2021) further showed that this ideal is also a maximal ideal of τ_f. Note that the axioms for an ideal of a semiring only consider the addition of a finite number of elements of the ideal to be an element of the ideal itself. However, it is also possible to show that for a primal space with a connected component C composed of finite orbits, that the arbitrary union of elements of the ideal $\psi(C)$ is also an element of the ideal. For that, consider the collection $\{A \in \tau_f : A \in \psi(C)\}$. If we assume $\bigcup_{A \in \psi(C)} A \supseteq C$ then applying Lemma 2.6 and Lemma 3.4 we have that there exists an open set A_0 from the given collection such that $C \subseteq A_0$, a contradiction. It must hold then that $\bigcup_{A \in \psi(C)} A \not\supset C$, and therefore $\bigcup_{A \in \psi(C)} A \in \psi(C)$.

Theorem 4.3. *Let (X, τ_f) be a primal space and G the orbit of a point, then $\phi(G) := \{A \in \tau_f : A \cap G = \emptyset\}$ is a prime ideal of τ_f.*

Proof: Let $A, B \in \phi(G)$ and $V \in \tau_f$. Since $G \cap A = \emptyset$ and $G \cap B = \emptyset$ then $G \cap (A \cup B) = \emptyset$, which implies $(A \cup B) \in \phi(G)$. It is trivial to see that $A \cap V \in \phi(G)$, moreover $X \notin \phi(G)$ given that $X \cap G \neq \emptyset$. It also holds that if $P \cap Q \in \phi(G)$ for some $P, Q \in \tau_f$ then $P \in \phi(G)$ or $Q \in \phi(G)$. Indeed, let $p \in (P \cap G)$ and $q \in (Q \cap G)$, if $p = q$ then $p \in (P \cap Q) \cap G = \emptyset$, a contradiction. On the other hand, and without loss of generality, we assume $p \neq q$ and $q \in cl(p)$, then ker (q) contains p and $\ker(q) \subset Q$. Therefore $p, q \in Q$ which implies $(P \cap Q) \notin \phi(G)$, a contradiction. It follows that $P \in \phi(G)$ or $Q \in \phi(G)$.

Theorem 4.4. *Let (X, τ_f) be a primal space and G a subset of a periodic orbit F, then $\phi(G) \coloneqq \{A \in \tau_f : A \cap G = \emptyset\}$ is a prime ideal of τ_f.*

Proof: Let $A, B \in \phi(G)$ and $V \in \tau_f$. Since $G \cap A = \emptyset$ and $G \cap B = \emptyset$ then $G(A \cup B) = \emptyset$, which implies $A \cup B \in \phi(G)$. It is clear that $A \cap V \in \phi(G)$, moreover $X \notin \phi(G)$ since $X \cap G \neq \emptyset$. Finally, if $P \cap Q \in \phi(G)$ then $P \in \phi(G)$ or $Q \in \phi(G)$. Indeed, if we assume $p \in (P \cap G)$ and $q \in (Q \cap G)$ where p, q are two distinct points of G, then we have that $q \in \ker(q) \subset Q$, and given that F is a periodic orbit, then $p \in$ ker (q). Therefore $p, q \in Q$, a contradiction. It follows that $P \in \phi(G)$ or $Q \in \phi(G)$.

As with the previous ideal, it is possible to show that the arbitrary union of elements of the ideal $\phi(F)$ is also an element of the ideal itself.

Proposition 4.1. *Let (X, τ_f) be a primal space, F a subset of X and $\phi(F) \coloneqq \{A \in \tau_f : A \cap F = \emptyset\}$, then $\bigcup_{A \in \phi(F)} A \in \phi(F)$.*

Proof: If we assume $\bigcup_{A \in \phi(F)} A \notin \phi(F)$, then $\left(\bigcup_{A \in \phi(F)} A\right) \cap F \neq \emptyset$, therefore there exists $x \in X$ such that $x \in \bigcup_{A \in \phi(F)} A$ and $x \in F$. Then it must hold that $x \in A_0$ for some $A_0 \in \phi(F)$. This implies $A_0 \cap F \neq \emptyset$, a contradiction.

We now present some results regarding the primal topology induced on $\mathbb{R}^n$ by a linear transformation $f: \mathbb{R}^n \to \mathbb{R}^n$. Recall that any linear transformation can be written in matrix form, that is, if f is a linear transformation, then it can be written as $f(x) = Ax$ with A a square $n \times n$ matrix and $x \in \mathbb{R}^n$. For instance, the linear transformation $Rot: \mathbb{R}^2 \to \mathbb{R}^2$ associated with vector rotation in $\mathbb{R}^2$ can be written as follows:

$$Rot(v) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot v$$

With $v \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$. This linear transformation induces a primal topology on $\mathbb{R}^2$, and it can be seen that regardless of the choice of θ, the primal space $(\mathbb{R}^2, \tau_{Rot})$ will have an infinite number of connected components. This makes $\mathbb{R}^2$ a non-connected topological space, characteristic that is not obtained when $\mathbb{R}^2$ is equipped with the usual topology. Additionally, this transformation is invertible, given that the associated matrix is invertible. Other interesting results arise from considering linear transformations associated with real entries matrices as the following:

Example 4.2. *Consider the following diagonal $n \times n$ matrix*

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \cdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

The function $f: \mathbb{R}^n \to \mathbb{R}^n$ defined as $f = Av$ with $v \in \mathbb{R}^n$ induces a primal topology on $\mathbb{R}^n$, which can be denoted by τ_A.

It is evident that the open sets of this space will strongly depend on the values of the diagonal. If, for instance, one of the diagonal values $\lambda_i = 0$, then $\ker(\{0_{\mathbb{R}^n}\})$ would be an infinite set given that every vector $u \in \mathbb{R}^n$ with all coordinates equal to 0 except for the i-th coordinate will satisfy the equality $Au = 0_{\mathbb{R}^n}$. Note as well that in this case the matrix will not be invertible, and so the linear transformation is not invertible. On the other hand, if all values on the diagonal are distinct from zero, the determinant of the matrix A is not zero and the matrix is invertible. In this case the following holds.

Lemma 4.4. *Let $A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \cdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ be a diagonal $n \times n$ matrix. If every $\lambda_i \neq 0$ then $ker(\{0_{\mathbb{R}^n}\}) = \{0_{\mathbb{R}^n}\}$ and $\{0_{\mathbb{R}^n}\} \in \tau_A$.*

Proof: If every $\lambda_i \neq 0$, then for the system $Ax = 0_{\mathbb{R}^n}$ we have $\lambda_i x_i = 0$, then $x_i = 0$ for $i = 1,2, \dots, n$ which implies $x = 0_{\mathbb{R}^n}$. Therefore $\ker(\{0_{\mathbb{R}^n}\}) = \{0_{\mathbb{R}^n}\}$ and $\{0_{\mathbb{R}^n}\} \in \tau_A$.

In general, the following result is obtained:

Corollary 4.1. *Let $A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \cdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ be a diagonal $n \times n$ matrix. If every $\lambda_i \neq 0$ then $\{0_{\mathbb{R}^n}\}$ is clopen.*

Note as well that $\mathbb{R}^n$ equipped with this topology is not a connected topological space. We may now consider arbitrary linear transformations associated with a matrix A of real entries.

Lemma 4.5. *If A is an invertible matrix and $U \in \tau_A$, then $U \setminus \{0_{\mathbb{R}^n}\} \in \tau_A$.*

Proof: We have $A^{-1}(U \setminus \{0_{\mathbb{R}^n}\}) = A^{-1}(U) \setminus \{0_{\mathbb{R}^n}\}$. But since $U \in \tau_A$ i.e., $A^{-1}(U) \subset U$, then $A^{-1}(U) \setminus \{0_{\mathbb{R}^n}\} \subset (U \setminus \{0_{\mathbb{R}^n}\})$ which implies $A^{-1}(U \setminus \{0_{\mathbb{R}^n}\}) \subset (U \setminus \{0_{\mathbb{R}^n}\})$.

Lemma 4.6. *If A is an invertible matrix, then $M \coloneqq \{U \in \tau_A : 0_{\mathbb{R}^n} \notin U\}$ is an ideal of τ_A.*

Proof: $0_{\mathbb{R}^n} \notin \emptyset$, then $\emptyset \in M$. Let $U, V \in M$, then $U, V \in \tau_A$, therefore $(U \cup V) \in \tau_A$. Moreover, $0_{\mathbb{R}^n} \notin (U \cup V)$, given that $0_{\mathbb{R}^n} \notin U$ and $0_{\mathbb{R}^n} \notin V$. If $U \in M$ and $W \in \tau_A$, then $(U \cap W) \in \tau_A$. Moreover $0_{\mathbb{R}^n} \notin (U \cap W)$ since $0_{\mathbb{R}^n} \notin U$, then $(U \cap W) \in M$.

Theorem 4.5. *If A is an invertible matrix, then $M \coloneqq \{U \in \tau_A : 0_{\mathbb{R}^n} \notin U\}$ is a maximal ideal of τ_A.*

Proof: If N is an ideal of τ_A such that $M \subsetneq N$ then there exists $V \in N$ such that $0_{\mathbb{R}^n} \in V$. In particular we have that $\{0_{\mathbb{R}^n}\} = (\{0_{\mathbb{R}^n}\} \cap V) \in N$. By Lemma 4.5 we have that $(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \in \tau_A$ and $(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \in M \subset N$. Therefore $\mathbb{R}^n = [(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \cup \{0_{\mathbb{R}^n}\}] \in N$, then $N = \tau_A$ and M is maximal.

5. GROUP HOMOMORPHISMS AND PRIMAL TOPOLOGIES

In this section we provide some generalizations to some results shown by Garcia-Mendoza et al (2021). More specifically, we generalize the notion of primal topologies induced on finite dimension vector spaces.

Theorem 5.1. *If the function $f: X \to X$ that defines the primal space (X, τ_f) is the identity function, then the space is zero-dimensional.*

Proof: If f is the identity function, then τ_f is the discrete topology and for each $x \in X$ we have $cl(x) = \{x\}$. From Lemma 3.3 by Garcia-Mendoza et al (2021), we have $\phi(\{x\})$ is a maximal ideal. In order to prove this Lemma, it is enough to show that any given prime ideal is also a maximal ideal. We prove this by showing that for a given prime ideal P of τ_f there exists $y \in X$ such that $y \notin P$ and $\phi(\{y\}) = P$. Let $A \in \phi(\{y\})$, then $A \cap \{y\} = \emptyset \in P$. Given that $\{y\} \notin P$ then it must hold $A \in P$ and $\phi(\{y\}) \subset P \subset \tau_f$. Since $\phi(\{y\})$ is maximal, then $\phi(\{y\}) = P$.

Definition 5.1. *Let $F: G \to H$ be a group homomorphism. The set $KER(F) \coloneqq \{x \in G : F(x) = 0_H\}$ is called the algebraic kernel of F, where 0_H is the identity element of the group H.*

It is a well-known result that if $F: G \to H$ is a group homomorphism then F is 1-1 if and only if $KER(F) = \{0_G\}$. We provide a topological characterization of 1-1 group homomorphisms.

Theorem 5.2. *Let G be a group and $F: G \to G$ a group homomorphism. Then F is 1-1 if and only if $\{0\} \in \tau_F$.*

Proof: Given that F is a group isomorphism we have $KER(F) = \{0\}$, that is, $\{x \in G: F(x) = 0\} = \{0\}$, which means $F^{-1}(\{0\}) = \{0\}$ and $\{0\} \in \tau_F$. On the other hand, if $\{0\} \in \tau_F$ then $KER(F) = F^{-1}(\{0\}) \subset \{0\}$, given that $\{0\} \in \tau_F$, then $\{0\} \subset \ker(\{0\}) \subset \{0\}$, which implies $KER(F) = \{0\}$ and F is 1-1.

Corollary 5.1. *If G is a group anf $F: G \to G$ is an injective group homomorphism then $\{0\}$ is clopen.*

Proof: By the previous Theorem we have that $\{0\}$ is open. Given that $F(\{0\}) = \{0\}$ then $\{0\}$ is also closed.

Theorem 5.3. *Let $T: G \to G$ be a group homomorphism and P a prime ideal of τ_T. If $\{0\} \notin P$ then P is maximal and $P = \phi(0)$.*

Proof: Given that T is a group homomorphism then $T(0) = 0$, which is equivalent to $cl(0) = 0$, and by Lemma 3.3 by Garcia-Mendoza et al (2021) we have $\phi(0)$ is a maximal ideal of τ_T. Let $A \in \phi(0)$, then $0 \notin A$, which implies $\{0\} \cap A = \emptyset \in P$. Given that P is a prime ideal of τ_T and $\{0\} \notin P$ then $A \in P$. Therefore $\phi(\{0\}) \subset P$ and given that $\phi(\{0\})$ is maximal we have $\phi(\{0\}) = P$.

6. CONCLUDING REMARKS

In this paper we explored some of the properties of primal topologies seen as semirings. Some of the advantages of studying the algebraic properties of these topologies is reflected, for example, in Theorem 4.5. where it was possible to construct another algebraic condition for the invertibility of a matrix considering the topological and algebraic properties of the primal topology induced on $\mathbb{R}^n$ by the matrix. Moreover, considering topologies as semirings has opened the door to address certain problems in a novel way. For instance, the Collatz conjecture, a problem that has not been solved yet, can be studied from a topological point of view, given that the topology induced on $\mathbb{N}$ is a primal topology. In this paper, some algebraic structures such as ideals, maximal and prime ideals are considered, which could shed light on the study of the conjecture.

7. DISCLOSURE OF CONFLICT OF INTEREST OF THE AUTHORS

The authors declare no conflict of interest.

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Authors' contribution

Author	Contribution
Carlos Garcia-Mendoza	Writing of manuscript, bibliographic search, proofs
Jorge Enrique Vielma	Proposal of main theorems, manuscript structure
José Játem	Proposal of generalizing theorems, manuscript proof reading