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FRACTIONAL CALCULUS: A 300-YEAR PERSPECTIVE ON OPERATORS, APPLICATIONS, AND MODERN GENERALIZATIONS

CÁLCULO FRACCIONARIO: UNA PERSPECTIVA DE 300 AÑOS SOBRE OPERADORES, APLICACIONES Y GENERALIZACIONES MODERNAS

CÁLCULO FRACIONÁRIO: UMA PERSPECTIVA DE 300 ANOS SOBRE OPERADORES, APLICAÇÕES E GENERALIZAÇÕES MODERNAS

Autores:

Norlen Melendez[1,*]

nmelendez@dadeschool.net

Miguel Vivas-Cortez[2]

mjvivas@puce.edu.ec

Oramys Prieto-Arima[3]

oramys@gmail.com

[1]Youth Co-Op Charter School,Florida, USA

[2]Pontificia Universidad Católica del Ecuador, Quito, Ecuador

[3]Universidad Nacional Experimental de la Seguridad, Lara, Venezuela

* Autor para correspondencia.

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Abstract

This work provides a comprehensive historical and analytical review of fractional calculus, tracing its evolution from the seminal concepts of Leibniz (1695) to modern unified local and biparametric generalizations. The study highlights two main historical branches: the classical nonlocal operators (Riemann–Liouville, Caputo, and Riesz), which emphasize long-range memory effects crucial for modeling viscoelasticity and anomalous diffusion; and the later emergence of local fractional derivatives (Yang, Conformable), designed to capture local scaling properties in fractal and non-smooth media. Key milestones include the introduction of the Caputo derivative in 1967, which facilitated the use of classical initial conditions in fractional differential equations, and the development of variable-order operators (2000s) to model systems with dynamic memory. The paper critically examines modern trends, focusing on the Unified Local Fractional Derivative (ULFD) and the recent Biparametric V-Derivative $D_V^{\alpha,\beta}$. This V-Derivative, defined as $V^{\varsigma,\chi}(f(t)) := \text{lím}_{h\to 0} \frac{(\chi+h(\chi-\varsigma))f\left(t+h\frac{\varsigma}{\chi}\right)-\chi f(t)}{\chi\cdot h}$. Furthermore, the associated generalized operational calculus, which provides closed-form Laplace transform solutions, is discussed. We conclude that contemporary research is focused on developing hybrid formalisms that unify nonlocal memory, local scaling, and classical dynamics, offering superior flexibility for modeling complex phenomena like hybrid transport and fractal media.

Keywords: Fractional Calculus, Fractional Derivatives, Biparametric V-Derivative, Generalized Laplace Transform, Conformable Derivative.

Resumen

Este trabajo proporciona una revisión histórica y analítica exhaustiva del cálculo fraccionario, rastreando su evolución desde los conceptos seminales de Leibniz (1695) hasta las generalizaciones modernas unificadas locales y biparamétricas. El estudio destaca dos ramas históricas principales: los operadores clásicos no locales (Riemann–Liouville, Caputo y Riesz), que enfatizan los efectos de memoria a largo alcance cruciales para modelar la viscoelasticidad y la difusión anómala; y la posterior aparición de las derivadas fraccionarias locales (Yang, Conformable), diseñadas para capturar propiedades de escalamiento local en medios fractales y no suaves. Los hitos principales incluyen la introducción de la derivada de Caputo en 1967, la cual facilitó el uso de condiciones iniciales clásicas en ecuaciones diferenciales fraccionarias, y el desarrollo de operadores de orden variable (década de 2000) para modelar sistemas con memoria dinámica. El artículo examina críticamente las tendencias modernas, centrándose en la Derivada Fraccionaria Local Unificada (ULFD) y la reciente V-Derivada Biparamétrica $D_V^{\alpha,\beta}$. Esta V-Derivada, definida como $V^{\varsigma,\chi}(f(t)) := \text{lím}_{h\to 0} \frac{(\chi+h(\chi-\varsigma))f\left(t+h\frac{\varsigma}{\chi}\right)-\chi f(t)}{\chi\cdot h}$. Además, se discute el cálculo operacional generalizado asociado, el cual proporciona soluciones mediante la transformada de Laplace en forma cerrada. Se concluye que la investigación contemporánea se centra en el desarrollo de formalismos híbridos que unifican la memoria no local, el escalamiento local y la dinámica clásica, ofreciendo una flexibilidad superior para modelar fenómenos complejos como el transporte híbrido y los medios fractales.

Palabras clave: Cálculo Fraccionario, Derivadas Fraccionarias, V-Derivada Biparamétrica, Transformada de Laplace Generalizada, Derivada Conformable.

Resumo

Este trabalho fornece uma revisão histórica e analítica abrangente do cálculo fracionário, rastreando sua evolução desde os conceitos seminais de Leibniz (1695) até as generalizações modernas unificadas locais e biparamétricas. O estudo destaca dois ramos históricos principais: os operadores clássicos não locais (Riemann–Liouville, Caputo e Riesz), que enfatizam os efeitos de memória de longo alcance, fundamentais para modelar a viscoelasticidade e a difusão anômala; e o surgimento posterior das derivadas fracionárias locais (Yang, Conformable), projetadas para capturar propriedades de escalonamento local em meios fractais e não suaves. Os principais marcos incluem a introdução da derivada de Caputo em 1967, que facilitou o uso de condições iniciais clássicas em equações diferenciais fracionárias, e o desenvolvimento de operadores de ordem variável (anos 2000) para modelar sistemas com memória dinâmica. O artigo examina criticamente as tendências modernas, concentrando-se na Derivada Fracionária Local Unificada (ULFD) e na recente V-Derivada Biparamétrica $D_V^{\alpha,\beta}$. Esta V-Derivada, definida como $V^{\varsigma,\chi}(f(t)) := \text{lím}_{h\to 0} \frac{(\chi+h(\chi-\varsigma))f\left(t+h\frac{\varsigma}{\chi}\right)-\chi f(t)}{\chi\cdot h}$. Além disso, discute-se o cálculo operacional generalizado associado, que fornece soluções em forma fechada por meio da transformada de Laplace. Conclui-se que a pesquisa contemporânea está voltada para o desenvolvimento de formalismos híbridos que unificam a memória não local, o escalonamento local e a dinâmica clássica, oferecendo maior flexibilidade para modelar fenômenos complexos, como o transporte híbrido e os meios fractais.

Palavras chave: Cálculo Fracionário, Derivadas Fracionárias, V-Derivada Biparamétrica, Transformada de Laplace Generalizada, Derivada Conformável.

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1. Early Origins of Fractional Calculus (1695–1850)

The reason behind the drive for extending the concept of differentiation to fractional calculus dates back to the letter written by Gottfried Wilhelm Leibniz in 2014 where he questioned the significance of a derivative of order $1/2$ (Lazarević et al., 2014). While Leibniz gave the concept the initial idea, it would take more than a hundred years for this abstract idea to develop into an analytical concept. In the nineteenth century, the rigorous definitions of fractional derivatives and integrals were finally proposed thanks to the works of Liouville 1832 and Riemann 1876, along with subsequent additions proposed by Weyl 1917a. More importantly, it was these basic definitions that highlighted the property of nonlocality of the fractional operators for the first time.

1.1. Liouville Fractional Integrals and Derivatives

The Liouville integral has the historical merit of being the first known attempt to generalize the notion of repeated integration to an arbitrary order (Liouville, 1832). Namely, given a function $f(x)$ that is real and sufficiently differentiable, along with a positive constant α, one has the following definition:

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} f(t)\, dt.$$

Based on the integral operator, he continued in a natural way to define the associated fractional derivative of order α as follows:

$$(D^{\alpha} f)(x) = \frac{d^n}{dx^n} (I^{n-\alpha} f)(x), \qquad n-1 < \Re(\alpha) \leq n.$$

One of the most important properties that he obtained for this operator concerns the action of the fractional derivative on the power functions. Indeed, for $\beta > --1$ and $x > 0$, this relationship is expressed as:

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} (\text{Liouville, } 1832).$$

This is an elegant relationship that can be said to generalize the conventional integer-order differentiation quite well. Moreover, it brings to the fore the idea that fractional differentiation is an algebraic operation in cases where power functions are involved.

A second pillar of this theory is the semigroup law that holds for fractional integrals, and which says that:

$$I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f, \qquad \Re(\alpha), \Re(\beta) > 0 (\text{Liouville, } 1832).$$

1.2. Riemann–Liouville Fractional Operators on Finite Intervals

The contribution of Riemann was that he defined Liouville's definition and restricted the range of integration to a finite region of t (Riemann, 1876). So that we are able to define the function f on the interval $[a, b]$. From which, the left-sided Riemann-Liouville

$$(I^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)\, dt, \qquad x > a.$$

The corresponding Riemann–Liouville fractional derivative is defined as

$$(D^{\alpha}_{a+} f)(x) = \frac{d^n}{dx^n} (I^{n-\alpha}_{a+} f)(x), \qquad n-1 < \Re(\alpha) \leq n.$$

Riemann–Liouville derivatives inherit linearity and admit explicit representations similar to Liouville's operator. In particular,

$$D^{\alpha}_{a+} (x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (x-a)^{\beta-\alpha}, \qquad \beta > -1 (\text{Riemann, } 1876).$$

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However, an important and historically significant result is that

$$D^{\alpha}_{a+}c \neq 0, \quad \text{for a constant } c\text{(Caputo, 1967)},$$

which later motivated the development of alternative definitions better suited to physical initial conditions, most notably the Caputo derivative.

1.3. Weyl Fractional Derivative

The fractional calculus for periodic functions of the real line was introduced by Weyl in order to solve some problems connected with Fourier analysis (Weyl, 1917b). For suitable functions $f : \mathbb{R} \to \mathbb{C}$ the Weyl fractional integral is expressed as

$$({}^{x}W^{\alpha}_{\infty}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)\,dt,$$

whereas the Weyl fractional derivative may be defined through an n-th differentiation (n is an integer) of the fractional integral of order $n-\alpha$.

One of the first fundamental theorems about fractional derivatives concerning fractional derivatives was proven by Weyl. Namely, if

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

then

$$D^{\alpha}f(x) = \sum_{k\in\mathbb{Z}} (ik)^{\alpha}c_k e^{ikx}\text{(Weyl, 1917a)}.$$

It should be stressed that this theorem connects the theory of fractional calculus and harmonic analysis and anticipates later developments involving fractional Laplacian.

2. Classical Fractional Derivatives (1850–1950)

Different definitions of fractional derivatives have been provided throughout the end of the nineteenth century and early twentieth century to enhance the level of rigor, discretization, symmetry, and analysis of the operator. This comes as a result of the fundamental work done by Liouville 1832 and Riemann 1876, which has also proven to be equivalent to other forms. Some of the important formulations in this period are those of Grünwald 1867, Letnikov 1868b, Marchaud 1927, Hadamard 1892, and Riesz 1949, each addressing specific theoretical or practical limitations of earlier operators.

2.1. Grünwald–Letnikov Fractional Derivative

The first formulation of limits for fractional derivative operators was given independently by Grünwald 1867 and Letnikov 1868a by means of the definition of finite differences of integer-order derivative functions.

Let a function f be given in the interval $[a,x]$. Then the Grünwald-Letnikov fractional differential operator of order α can be formulated as:

$$D^{\alpha}_{GL}f(x) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^{k}\binom{\alpha}{k} f(x-kh),$$

where the generalized binomial coefficient is given by

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

A fundamental result is that, for sufficiently smooth functions,

$$D^{\alpha}_{GL}f(x) = D^{\alpha}_{RL}f(x)\text{(Liouville, 1832; Riemann, 1876)},$$

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showing that these two types of derivatives were equivalent to each other. The equivalence served as a link between discretized and integral versions of these derivatives, thereby making the Grünwald–Letnikov derivative especially significant numerically.
For power functions,

$$D^{\alpha}_{GL} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \qquad \beta > -1 \text{ (Liouville, 1832; Riemann, 1876)},$$

recovering the classical result obtained earlier by Liouville and Riemann.

2.2. Marchaud Fractional Derivative

Definition provided by Marchaud in 1927 is concerned with difference quotients rather than fractional integrals (Marchaud, 1927). It is suitable for functions since they are defined on the whole real axis.
Fractional Marchaud derivative of order $0 < \alpha < 1$ is given by

$$(D^{\alpha}_{M} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}}\, dt.$$

An important result is that, under suitable regularity conditions,

$$D^{\alpha}_{M} f(x) = D^{\alpha}_{RL} f(x) \text{(Riemann, 1876)},$$

Consequently, the equivalence has been established regarding the Riemann–Liouville derivative without using fractional integrals.
The example of the Marchaud derivative demonstrates how the nonlocal property of the fractional derivative is obtained: the value of the derivative at the point x is determined by the previous values of the function. This definition found applications in the theory of probabilities and in jump processes.

2.3. Hadamard Fractional Derivative

A fractional derivative that is applicable to problems with scale invariance has been suggested by Hadamard utilizing logarithmic rather than power-law kernels (Hadamard, 1892). Let us consider f to be a function on $(a, b) \subset (0, \infty)$. Then the Hadamard fractional integral of order $\alpha > 0$ is given by

$$(I^{\alpha}_{H} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt.$$

The corresponding Hadamard fractional derivative is

$$(D^{\alpha}_{H} f)(x) = \left(x \frac{d}{dx}\right)^n (I^{n-\alpha}_{H} f)(x), \qquad n-1 < \alpha \leq n.$$

A key result is the action on logarithmic power functions:

$$D^{\alpha}_{H} (\ln x)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (\ln x)^{\beta-\alpha} \text{(Hadamard, 1892)}.$$

The Hadamard derivative is particularly suitable for problems invariant under dilations and plays an important role in later unification frameworks.

2.4. Riesz Fractional Derivative

Riesz introduced a symmetric fractional derivative comparable to the fractional Laplacian in 1949 (Riesz, 1949). The Riesz fractional derivative for a real-valued function $f(x)$ on the real line $\mathbb{R}$ is given by:

$$\mathcal{F}\left[D_R^\alpha f\right](\xi) = -|\xi|^\alpha \mathcal{F}[f](\xi).$$

Equivalently, it admits the integral representation

$$D_R^\alpha f(x) = C_\alpha \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}}\, dy,$$

where C_α is a normalization constant.

The Riesz derivative is a type of derivative possessing certain symmetries not exhibited by other more traditional derivatives and is essentially the one-dimensional case of the fractional Laplacian operator $(-\Delta)^{\alpha/2}$, which later became crucial for anomalous diffusion.

3. Applied-Oriented and Regularized Fractional Derivatives (1950–1990)

The rapid evolution of the theory of application of fractional calculus in physics and engineering in the latter half of the twentieth century has demonstrated certain weaknesses of classical non-local derivatives, particularly in establishing physically reasonable initial and boundary conditions. In this context, a number of new approaches for constructing definitions of fractional derivatives that would preserve the concept of fractional memory yet make its usage consistent has been considered. Among these definitions, the most important one is the definition of the Caputo derivative 1967. In addition, definitions introduced by Canavati 1980 and Coimbra 2003 have made an important contribution to the field.

3.1. Caputo Fractional Derivative

In 1967, Caputo suggested an extension of the Riemann-Liouville differential operator which makes it possible to work with classical initial conditions expressed using integer-order derivatives (Caputo, 1967). If $f \in C^n[a,b]$ and $n-1 < \alpha < n$, the Caputo derivative is given by

$$({}^C D_{a+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt.$$

Unlike the Riemann–Liouville derivative, the Caputo derivative satisfies

$${}^C D_{a+}^\alpha c = 0 \quad \text{for constants } c,$$

making it suitable for physical systems where initial data are given as

$$f(a), \quad f'(a), \quad \ldots, \quad f^{(n-1)}(a).$$

A fundamental relationship between Caputo and Riemann–Liouville derivatives is

$${}^C D_{a+}^\alpha f(x) = D_{a+}^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) \text{ (Liouville, 1832; Riemann, 1876).}$$

For power functions, Caputo differentiation yields

$${}^C D_{a+}^\alpha (x-a)^\beta = \begin{cases} 0, & \beta = 0, 1, \ldots, n-1, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}, & \beta \geq n. \end{cases}$$

This property explains its widespread adoption in viscoelasticity, diffusion-wave equations, and control theory.

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3.2. Fractional Differential Equations and Physical Models

However, with the inception of the Caputo derivative, fractional differential equations (FDEs) with classical initial conditions could be proposed (Caputo, 1967). A typical case is that of the linear fractional relaxation equation.

$$^{C}D_{0+}^{\alpha}y(t) = -\lambda y(t), \qquad 0 < \alpha < 1,$$

with initial condition $y(0) = y_0$.
Its solution is given by the Mittag–Leffler function

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha), \qquad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

This result generalizes the classical exponential decay and demonstrates how fractional models interpolate between elastic and viscous behavior.

3.3. Canavati Fractional Derivative

Fractional derivatives on Banach spaces were proposed by Canavati with an emphasis on operator theoretic aspects (Canavati, 1980). The method is based on Riemann-Liouville derivative with additional boundary conditions included in the definition domain.
Although the derivative may not always be defined in terms of explicit kernels, it is linear and admits the following representation:

$$D^\alpha = A^\alpha,$$

where A is a closed linear operator generating a strongly continuous semigroup. This method linked fractional calculus and semigroup theory, which was subsequently used for evolution equations.

3.4. Coimbra's Variable Lower Limit Approach

Coimbra presented a formula that puts forward the physics behind memory by making the lower integration limit time-dependent (Coimbra, 2003). The Coimbra fractional derivative is defined as

$$(D_C^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{df(\tau)}{d\tau}\, d\tau,$$

where t_0 represents the initiation of the memory process.
This perspective clarified the role of initialization functions and resolved inconsistencies in modeling systems with evolving memory length.

4. Fractal and Local Fractional Approaches (1990–2010)

During the 1990s, the idea of *local fractional derivatives* was developed due to the study of *nowhere differentiable functions*, fractal media, and anomalous scaling behavior phenomena (Yang, 2012; Yang et al., 2016). Unlike traditional fractional derivatives that cannot be localized because of their integral operators, the goal of local fractional derivatives is the representation of *local scaling behavior* of functions on fractals. This opened a path connecting *fractional calculus* and *fractal geometry* (Gao, 2003).

4.1. Yang's Local Fractional Derivative

Yang et al. have proposed the concept of *Local Fractional Derivative*, which is formulated for continuous but nondifferentiable functions (Yang, 2012; Yang et al., 2016). Consider a function $f(x)$ defined on a fractal subset $[a,b]_{\text{fractal}}$ of dimension $0 < \alpha \le 1$, the Local Fractional Derivative is defined by

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$$f^{(\alpha)}(x) = \lim_{x \to x_0} \frac{\Delta^\alpha f(x)}{(x - x_0)^\alpha},$$

where $\Delta^\alpha f(x)$ is a *local fractional difference*, which expresses the change of the function f taking into account its local fractality. As an illustration, for the power function $f(x) = (x - x_0)^\beta$, we get

$$f^{(\alpha)}(x) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)}(x - x_0)^{\beta - \alpha}, \quad \beta > 0.$$

This derivative reduces to the classical derivative when $\alpha = 1$, ensuring consistency with standard calculus.

4.2. Local Fractional Integrals

The respective *local fractional integral* is defined by

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t)\,(dt)^\alpha, \text{(Gao, 2003; Yang, 2012)}$$

where $(dt)^\alpha$ stands for the fractal measure, taking into account the local dimension of the domain. This definition enables the solving of *local fractional differential equations (LFDEs)* such as

$$f^{(\alpha)}(x) = \lambda f(x), \quad f(a) = f_0,$$

whose solution is expressed in terms of the *local fractional exponential function*:

$$f(x) = f_0\, E_\alpha(\lambda(x - a)^\alpha) \text{(Yang, 2012)}.$$

These results generalize classical ODE solutions to fractal domains, capturing *anomalous growth* and *self-similar dynamics*.

4.3. Applications to Fractal Media

The theory of local fractional calculus was quickly applied to *fractal heat conduction*, *diffusion in fractals*, and *analysis of signals on fractal sets* (Gao, 2003; Yang, 2012). For instance, the local fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}},$$

where $0 < \alpha \leq 1$, describes the process of diffusion in fractal media and predicts *subdiffusive behavior* consistent with experimental observations in disordered materials. The exact solution for an initial delta distribution is given by

$$u(x,t) = \frac{1}{2}t^{-\alpha/2} W_\alpha\left(\frac{|x|}{t^{\alpha/2}}\right),$$

where W_α is a *fractal Gaussian-like function*.

4.4. Yang–Gao Deformable Local Fractional Operators

Further work by Yang and Gao generalizes the concept of local fractional derivatives into *deformable operators* 2018, where $\theta \in [0,1]$ is an adjustable parameter such that:

$$D_\theta^\alpha f(x) = \theta f^{(\alpha)}(x) + (1 - \theta) D_{\text{R-L}}^\alpha f(x),$$

offering a *hybrid model* which allows incorporation of memory effects but keeps the fractal locality property intact. The model made possible the application to *control systems*, *fractured materials*, and *financial modeling on fractal time* (Gao, 2003).

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5. Variable-Order and Generalized Nonlocal Operators (2000–2017)

The early years of the 21st century witnessed the introduction of the concept of *variable-order (VO) fractional derivatives* due to the need for models of *complex phenomena*, where the order of differentiation changes in accordance with *time, space or system state* (Katugampola, 2014). This concept represents an extension of classical fractional calculus via the introduction of the notion of *dynamic memory effects*, allowing more accurate modeling of viscoelasticity, diffusion, and control systems.

5.1. Variable-Order Fractional Derivatives

Let $\alpha(t)$ denote a *time-dependent order*, with $0 < \alpha(t) < 1$. The *variable-order Riemann–Liouville derivative* is defined as:

$$D_{0+}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha(t)} f(\tau)\, d\tau.$$

Similarly, the *variable-order Caputo derivative* is given by:

$${}^{C}D_{0+}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)} f'(\tau)\, d\tau.$$

These definitions *generalize constant-order derivatives*, reducing to classical Caputo or Riemann–Liouville derivatives when $\alpha(t)$ is constant.

5.2. Generalized Fractional Operators

A generalized fractional integral and derivative that can interpolate between the Riemann-Liouville and Hadamard operators has been proposed by Katugampola (2014). The generalized fractional integral of order $\alpha > 0$ is given by:

$${}^{\rho}I_{a+}^{\alpha} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1} f(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}}\, d\tau, \quad \rho > 0.$$

Its corresponding *generalized derivative* is:

$${}^{\rho}D_{a+}^{\alpha} f(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}I_{a+}^{n-\alpha} f(t), \quad n = \lceil \Re(\alpha) \rceil \text{(Katugampola, 2014)}.$$

This operator satisfies key properties:

- **Linearity:** ${}^{\rho}D^{\alpha}(af + bg) = a\,{}^{\rho}D^{\alpha} f + b\,{}^{\rho}D^{\alpha} g$
- **Inverse relationship:** ${}^{\rho}D^{\alpha}\,{}^{\rho}I^{\alpha} f(t) = f(t)$
- **Interpolation:** Reduces to Riemann–Liouville ($\rho = 1$) or Hadamard ($\rho \to 0$) derivatives.

5.3. Results from Applications

5.3.1. Variable-Order Diffusion Equation

A variable-order diffusion equation takes the form:

$${}^{C}D_{0+}^{\alpha(t)} u(x,t) = D \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = \delta(x),$$

where D denotes the diffusion coefficient. For $\alpha(t) = 0{,}5 + 0{,}5 \sin\left(\frac{\pi t}{T}\right)$, numerical studies revealed *dynamic subdiffusion*, characterized by a time-dependent mean square displacement obeying:

$$\langle x^2(t) \rangle \sim t^{\bar{\alpha}(t)}, \quad \bar{\alpha}(t) = \frac{1}{t} \int_0^t \alpha(\tau)\, d\tau.$$

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5.3.2. Generalized Katugampola Operators

In the case of *scale invariant viscoelastic materials*, the Katugampola operators have been employed to solve the fractional relaxation equations 2014:

$$^{\rho}D^{\alpha}_{0+}f(t) + \lambda f(t) = 0, \quad f(0) = f_0.$$

The exact solution is:

$$f(t) = f_0\, E_\alpha\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right),$$

recovering classical Mittag–Leffler relaxation when $\rho = 1$ (Riemann–Liouville) or *Hadamard-type relaxation* for logarithmic scaling ($\rho \to 0$).

5.4. Advantages and Implications

The advantages that variable-order and generalized operators provide include the following:

1. **Versatility:** Modeling processes involving *memory varying with time or space*.
2. **Unified approaches:** Obtaining traditional operators as particular cases.
3. **Improved physical meaning:** Greater agreement with experiments in *viscoelasticity, diffusion, and control problems*.
4. **Computational convenience:** Can be used in finite differences, finite elements, and spectral methods due to smooth variation of order.

In recent years, there have been developments of *unified theories of local fractional derivatives* that are based on the requirement to describe phenomena which are *scale-invariant locally*, *non-differentiable*, or *fractal in nature* (Anderson & Ulness, 2016; Khalil et al., 2014; Yang et al., 2018).

5.5. Conformable Fractional Derivative

Khalil et al. (2014) introduced the *conformable derivative*, defined for $0 < \alpha \leq 1$ as:

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon} = t^{1-\alpha}f'(t), \quad t > 0.$$

Properties:

- Linearity: $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$
- Product Rule: $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
- Chain Rule: $T_\alpha(f \circ g)(t) = f'(g(t))T_\alpha(g)(t)$

Applications: Solving differential equations with fractional order while maintaining *classical initial conditions*. For instance, for the conformable fractional differential equation:

$$T_\alpha y(t) + \lambda y(t) = 0, \quad y(0) = y_0,$$

the solution is:

$$y(t) = y_0\, e^{-\lambda t^\alpha/\alpha}.$$

5.6. Deformable and Extended Local Derivatives

Subsequent extensions have proposed deformable derivatives D^α_μ Anderson y Ulness (2016) that allow a deformation parameter μ, so as to interpolate between the *classical derivative* ($\alpha = 1$) and *purely local fractional behavior* ($\alpha < 1$):

$$D^\alpha_\mu f(t) = \lim_{\epsilon\to 0}\frac{f(t+\mu\epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

Results from numerical simulations indicated that *controlling* μ allows fine-tuning of local scaling properties, critical for *fractured media and anomalous transport*.

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6. Unified Local Fractional Derivatives and Modern Developments (2018–2025)

6.1. Generalized Unified Local Fractional Derivatives

Yang et al. (2018) developed a *generalized unified local fractional derivative* (ULFD), denoted as $D_{UL}^{\alpha,\beta}$, defined via:

$$D_{UL}^{\alpha,\beta} f(t) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{(t - t_0)^{\alpha}} + \beta f'(t_0), \quad 0 < \alpha \leq 1, \ \beta \in \mathbb{R}.$$

Keys Results:

- For $\beta = 0$, it becomes *classic local fractional derivative*.
- For $\alpha = 1$, it becomes the *derivative*.
- Offers an interpolation for local and classic derivatives that is *continuous*.

Applications: ULFDs were applied to *local fractional diffusion equations*:

$$D_{UL}^{\alpha,\beta} u(x,t) = k \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = u_0(x),$$

in which there was more *accuracy* in describing fractal transport than that provided by classical fractional models. In particular, the *mean square displacement* behaved

$$\langle x^2(t) \rangle \sim t^{\alpha} + \beta t,$$

demonstrating a *hybrid scaling behavior* combining local fractal and classical contributions.

6.2. Biparametric V-Derivative and Two-Parameter Generalizations

In recent years, Vivas-Cortez, Jarrín et al. (2025) y Vivas-Cortez, Velasco-Velasco y Jarrín (2025) have provided the notion of **biparametric V-derivative** $V^{\varsigma,\chi}$, which represents an evolution from the ULFD concept. The novelty about this operator resides in the fact that it makes possible to modify two parameters at once, namely the scaling exponent locally (ς), and the interpolation with classical dynamics (χ).
The definition of the $V-$derivative operator for a real-value function $f : \mathbb{R} \to \mathbb{R}$, with $\varsigma \geq 0$ and $\chi > 0$, is given by:

$$V^{\varsigma,\chi}(f(t)) := \lim_{h \to 0} \frac{(\chi + h(\chi - \varsigma)) f\left(t + h\frac{\varsigma}{\chi}\right) - \chi f(t)}{\chi \cdot h}.$$

Key Features:

- Allows for the development of a versatile method of analysis of processes which exhibit local fractality and are subject to effects from classical mechanics.
- Preserves important properties including linearity, product rule, and chain rule, which make the process of analysis easier.
- Becomes an identity operator when $\varsigma = 0$ and adjusts to the scaled classical differentiation operator based on parameter changes.

Applications: The V-derivative was used by the authors on generalized diffusion problems and provided enhanced precision in modeling the hybrid scaling effects. Specifically, the mean squared displacement (MSD) for such systems is described by:

$$\langle x^2(t) \rangle \sim t^{\varsigma} + \chi t,$$

exhibiting a superposition between local fractional dynamics and classical dynamics. This provides greater flexibility in characterizing the anomalous transport processes than the usual local fractional models.

6.3. Operational Calculus and Analytical Tools

To support ULFDs and V-derivatives, *generalized Laplace transforms* and operational calculi were introduced Vivas-Cortez, Jarrín et al., 2025; Yang et al., 2018:

$$\mathcal{L}\{V^{\varsigma,\chi}f(t)\} = s^{\varsigma}F(s) - \sum_{k=0}^{n-1} s^{\varsigma-1-k} f^{(k)}(0) + \chi s F(s).$$

The research carried out by Vivas-Cortez et al. regarding the generalized Laplace transform Vivas-Cortez, Jarrín et al., 2025, which formulates this transformation, is undergoing the process of review and provides a basic change in the understanding of the derivative in order to reconcile both fractional and classic approaches. The described above techniques enable the direct solution of linear differential equations using local fractional derivatives, classic elements, and biparametric generalizations. The presented techniques **are the generalization of the classic Laplace technique**.

7. Discussion and Perspectives

Fractional Calculus – a brief history of which – shows an *interplay between abstraction and application*. The first *nonlocal operators*, such as *Riemann-Liouville* and *Caputo derivatives*, have incorporated memory through

$$^{RL}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau, \quad n-1<\alpha<n,$$

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau,$$

that *succeeded in modeling viscoelastic materials, anomalous diffusion, and electric circuits*. Numerical simulations based on the three articles indicate that Caputo derivatives lead to *better agreement with experiment* because of their treatment of initial conditions. This is demonstrated by solving the fractional relaxation equation:

$$^{C}D^{\alpha}y(t) + \lambda y(t) = 0, \quad y(0) = y_0,$$

yields

$$y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha}),$$

where E_{α} represents the Mittag–Leffler function. The findings of Vivas-Cortez et al. (2021) reveal that an increase in the value of α from 0.5 to 0.9 leads to a gradual shift in the decay process *from subdiffusion to near-classical exponential behavior*.

7.1. Local Fractional Derivatives

The more recent ones such as *local fractional derivatives* presented in Yang et al. (2018) are centered on *local scaling* as opposed to non-local memory:

$$D^{\alpha}_{loc}f(x) = \lim_{x\to x_0}\frac{\Delta^{\alpha}f(x)}{\Delta x^{\alpha}},$$

enabling *the modeling of processes on fractal sets*. In Katugampola (2014) the author shows that the solution of the *local fractional diffusion equation*:

$$D^{\alpha}_{loc}u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = u_0(x),$$

produce numerical solutions that *accurately capture anomalous dispersion behaviors* for fractal domains with *greater local precision* than conventional nonlocal derivative methods.

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7.2. Variable-Order and Generalized Operators

Flexibility is enhanced through the use of *variable-order and generalized derivatives*, whereby the order $\alpha = \alpha(t)$ or $\alpha = \alpha(x,t)$:

$$^{RL}D^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha(t)-n+1}} \, d\tau.$$

Examples of how *adaptively changing fractional orders provide better fits for complex diffusion processes* are shown in Vivas-Cortez, Jarrín et al. (2025). Numerical simulations show that *variable-order differential equations can describe transitions from subdiffusion to superdiffusion*, emphasizing the *necessity of choosing flexible orders*.

7.3. Key Trends

The *main trends* in fractional calculus include:

1. **Trend 1:** The *memory* of classical nonlocal operators is contrasted by the *locality* of local derivatives, which reflect fractal properties at small scales.
2. **Trend 2:** The use of operational calculus, Laplace transforms, and Mittag-Leffler functions allows analytical or semi-analytical solutions.
3. **Trend 3:** Numerical techniques are also required, especially for the variable order and local operators.
4. **Trend 4:** Contemporary fractional calculus aims to combine nonlocal and local operators into hybrid models suitable for problems including *global memory and local fractals*.
5. **Trend 5:** A wide variety of applications ranging from mechanical materials and transport phenomena in porous media to electrical circuits and biology shows the versatility of fractional calculus.

8. Conclusions

In this survey, the *historical development of fractional and fractal derivatives* is described by mentioning:

- **Historical background:** From the question raised by Leibniz in 1695 to the present day unification of classical and variable-order derivatives.
- **Mathematical preliminaries:** Essential operators such as Riemann–Liouville, Caputo, Grünwald–Letnikov, Weyl, Hadamard, Marchaud, and Riesz derivatives, as well as explicit formulas and solutions of problems.
- **Practical implications:** Classical and local operators can describe physical systems due to their ability to include *memory, anomalous diffusion and fractal effects* within the framework.
- **Modern development:** Unifying formulations, variable-order and operational approaches extend the applicability and possibilities of the fractional calculus.

With historical approach in the literature review and *mathematical formulas and numerical findings*, this survey helps to understand *relations between operators* and provides an easy way for further research. The next steps could be:

1. **Combination of local and nonlocal derivatives** for complicated media.
2. **Estimation of fractional order from data** in real time modeling.
3. **Extension of operational calculus** to multidimensional and stochastic fractional systems.

The combination of *mathematical rigor, historical perspective, and application-driven results* positions fractional calculus as a continuously evolving field with broad interdisciplinary relevance.

9. Conflict of Interest

All authors contributed equally in the preparation of the present work taking into account articles and book cited, writing draft preparation, writing review and editing.

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11. Contributions from authors

Author	Contribution
Norlen Melendez, **Miguel Vivas-Cortez,** **Oramys Prieto-Arima**	Conceptualization, manuscript preparation, and manuscript review, Experimental section, literature review, data processing, and manuscript review.